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Self-similar solution of the plane problem of the evolution of a hydraulic fracture crack in an elastic medium☆

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ABSTRACT

The plane problem of the evolution of a hydraulic fracture crack in an elastic medium is considered. It is established that a self-similar solution is only possible at a constant rate of fluid injection. The solution for the value of the crack opening is presented in the form of a series expansion in Chebyshev polynomials of the second kind, and expansion coefficients are obtained as a solution of the algebraic set of equations which arise when projecting the balance equation for injected fluid mass on Chebyshev polynomials. When there is no part of the region unfilled with fluid (a fluid lag), the gradient of the crack opening at the crack tip turns out to be singular when the finiteness of the medium stress intensity factor is taken into account. According to the estimate made, the rate of convergence of the series expansion for the solution in Chebyshev polynomials is fairly rapid for a small injection intensity.

Beginning with the pioneering paper by Zheltov and Kristianovich,[1] regularities of the evolution of the characteristics of a crack in various pumping regimes are the subject of a large body of research within the framework of various models. The balance equation for the injected fluid mass, which includes the flux gradient, forms the basis of the approach to the problem. In turn, the fluid flow in a given section of the crack is defined in terms of the crack opening and the pressure gradient along the crack. Existing theoretical models differ in the method of specifying the linear relation between the crack opening and the pressure; in some cases the relation between these quantities is local (the model of Perkins and Kern[2]), and in other cases this relation is nonlocal (the Spence–Sharp model [3]). In both models the problem of interest is to find exact self-similar solutions.

In local models the magnitude of the crack opening in every section has been found as a solution of the plane problem for an elastic space with a cut at a given pressure on both sides of the cut.[4] In this case the problem reduces to solving a partial differential equation of the non-linear heat conduction type. Self-similar solutions of such an equation are only obtained under certain special pumping conditions (power and exponential relations) and some solutions of this equation have been found.[4,5] In a recent paper by the author[7] these solutions were obtained as special solutions of the non-linear transfer equation of a more general type.

In the Spence–Sharp model the magnitude of the crack opening and the pressure are interconnected by a Hilbert transformation, obtained as a solution of the Riemann–Hilbert problem for an analytical function with a given values on the boundary of the complex plane. As a result, from the balance equation for the injected fluid mass one obtains a non-linear integro-differential equation that governs the laws of evolution of the crack characteristics. To solve this equation, it is best to use a series expansion of the opening magnitude in Chebyshev polynomials and to search for a time dependence of the expansion coefficients. A self-similar solution corresponds to the case when, for some forms of time dependence of the pumping rate, all the expansion coefficients depend on time in the same way. Recently,[3] it has been proposed that a series expansion of the solution in Chebyshev polynomials of the first kind should be employed, and a certain numerical procedure for finding the expansion coefficients within the framework of the requirement for a minimum of some functional at collocation points of limited number was outlined.

Below we propose another procedure for finding the expansion coefficients by obtaining a set of algebraic equations for these coefficients. Unlike the Spence–Sharp approach,[3] we use the representation for crack opening in the form of a series expansion in Chebyshev polynomials of the second kind, and this enables us to obtain a simple formula for the pressure in the form of a series expansion with known coefficients. Note also that the use of a series expansion in Chebyshev polynomials of arbitrary order is incorrect since the symmetry condition of the

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problem requires the use of an expansion only in polynomials of even order, and hence, the value found earlier for the first-order expansion coefficient[3] would seem to be incorrect (this value has to be equal to zero). The analysis carried out shows that a self-similar solution can only be obtained at a constant pumping rate, and the self-similar solutions obtained by Spence and Sharp with a power and exponential form of the pumping rate are not realizable. We also obtain formulae for the values of the initial crack opening and the length of the crack as functions of the initial volume of the injected fluid.

1. Statement of the problem

Consider the problem of the evolution of a hydraulic fracture crack in an unbounded elastic medium when an incompressible viscous fluid is pumped into the crack; the leakage of the fluid through the crack boundary assumed to be negligible. The basis for describing this phenomenon is the balance equation for the injected fluid mass. In the case of the plane problem this has the form

$$\frac{\partial H(x,t)}{\partial t}+\frac{\partial Q(x,t)}{\partial x} = q(x,t)$$

$$Q(x,t) = -\frac{1}{3\eta}H^3(x,t)\frac{\partial P(x,t)}{\partial x} \tag{1.1}$$

Here $H(x, t)$ is the magnitude of the crack opening, $q(x, t)$ is the density of the injection source, $Q(x, t)$ is the fluid flux in the section x, specified by Poiseuille's law, η is the viscosity, and $P(x, t)$ is the pressure.

According to the methods used when solving plane contact problems in the theory of elasticity,[8,9] in the static case the pressure and the magnitude of the crack opening are connected by the integral relation (the Hilbert transformation)

$$P(x,t) = -\frac{E}{\pi(1-\nu^2)}\int_{-R(t)}^{R(t)}\frac{\partial H(x',t)}{\partial x'}\frac{dx'}{x'-x}, \quad H(\pm R(t)) = 0 \tag{1.2}$$

where $2R(t)$ is the crack length, E is Young's modulus and ν is Poisson's ratio.

It should be noted that Eq. (1.2) holds when the whole volume of the crack is fully-filled with fluid. If near the crack tip there is a fluid-free region (a lag), the boundary pressure in this region is usually assumed to be constant (the so-called rock pressure). If there is a lag, one should consider a mixed boundary-value problem, and the relation between the opening magnitude and the pressure is not given by Eq. (1.2). Hence, the attempts by some authors (for example, see Ref. 10) to investigate the hydraulic fracture behaviour when there is a lag using relation (1.2) is erroneous. Below we consider analyze the case when there is no lag.

At the crack tip the fracture condition, according to which the stress intensity factor near the tip has to be equal to the stiffness of the medium K (Irwin's condition[11]), holds. Using the relation

$$\partial H/\partial x \sim -K/\sqrt{R^2(t)-x^2}$$

which holds near the crack tip, and the formula of the inverse Hilbert transformation for Eq. (1.2), Irwin's condition can be written as

$$\frac{\sqrt{R(t)}}{\pi E}\int_{-R(t)}^{R(t)}\frac{P(x,t)dx}{\sqrt{R(t)^2-x^2}} = K \tag{1.3}$$

From the three dimensional parameters K, E and η we can construct typical scales of length $L = K^2(1-\nu^2)/E^2$, time $T = 3\eta/E$ and pressure E. Now we change to new dimensionless variables by using the substitutions

$$x\to Lx, \quad t\to Tt, \quad H\to LH, \quad R\to LR, \quad P\to EP, \quad q\to\frac{L}{T}q$$

In the new dimensionless variables the set of equations takes the form

$$\frac{\partial H(x,t)}{\partial t}-\frac{\partial}{\partial x}\left(H^3(x,t)\frac{\partial P(x,t)}{\partial x}\right) = q(x,t) \tag{1.4}$$

$$P(x,t) = -\frac{1}{\pi}\int_{-R(t)}^{R(t)}\frac{\partial H(x',t)}{\partial x'}\frac{dx'}{x'-x} \tag{1.5}$$

$$\frac{\sqrt{R(t)}}{\pi}\int_{-R(t)}^{R(t)}\frac{P(x,t)dx}{\sqrt{R(t)^2-x^2}} = 1 \tag{1.6}$$

When searching for a solution of the problem the following statement is commonly used. Due to the system symmetry about the point $x = 0$, instead of the domain $-R(t) \le x \le R(t)$ we consider the domain $0 \le R(t)$ and instead of the pumping source density $q(x, t)$ we fix the flux at the symmetry point $Q(0, t) = Q_0(t)$. In this formulation we search for a self-similar solution and the form of function $Q_0(t)$ for which this solution can be realized. For the self-similar solutions known at present this occurs in the case of power ($Q_0(t) = At^{\alpha}$) and exponential ($Q_0(t) = Ae^{\alpha t}$) pumping laws.[3,5,6]

Self-similar solutions are constructed by using the similarity arguments (scale invariance), and the exponents of the power behaviour, which define the invariants of the group of scale transformations, are found from dimensionality considerations.

As applied to the problem under consideration, the approach presented below assumes that at the symmetry point $x = 0$ there is a point pumping source, i.e., $q(x, t) = q(t)\delta(x)$ and the pumping rate $q(t)$ determines the variation of injected fluid volume

$$q(t) = \frac{d}{dt}\int_{-R(t)}^{R(t)} H(x, t)dx \tag{1.7}$$

Eq. (1.5) defines the relation between the magnitude of the crack opening and the pressure, whereas Eq. (1.6) relates the pressure and the crack length. Note that these equations include time as a parameter, and the time-dependence of the crack characteristics is defined using the solution of the balance equation for the fluid mass (1.4). Thus, at the first stage, we must solve the equations of elasticity theory (1.5) and (1.6), and then search for the evolution law of the crack characteristics.

From Eq. (1.5) it can be seen that the pressure is expressed in terms of the gradient of the crack opening using the Hilbert transformation. In turn, by using the formula for the inverse Hilbert transformation, we can express the gradient of the opening in terms of the pressure

$$\frac{\partial H(x, t)}{\partial x} = \frac{1}{\pi\sqrt{R^2(t) - x^2}}\int_{-R(t)}^{R(t)} \frac{\sqrt{R^2(t) - x'^2}P(x', t)dx'}{x' - x} \tag{1.8}$$

From Eqs (1.5) and (1.8) it follows that the crack characteristics can be represented in the form

$$H(x, t) = \bar{H}(\xi, t), \ \frac{\partial H(x, t)}{\partial x} = \frac{1}{R(t)}\frac{\partial \bar{H}(\xi, t)}{\partial \xi}, \quad P(x, t) = \frac{\bar{P}(\dot{\xi} t)}{R(t)}, \quad \xi = \frac{x}{R(t)} \tag{1.9}$$

2. Solution of the equations of elasticity theory

We will represent the solution for the function $H(\xi, t)$, satisfying the boundary conditions $\bar{H}(\pm 1, t) = 0$, in the form of a series expansion in Chebyshev polynomials of the second kind $U_n(\xi)$ (Ref. 12)

$$\bar{H}(\xi, t) = H_0\sqrt{1 - \xi^2}\sum_k h_k(t)U_{2k}(\xi), \quad H_0 = \bar{H}(0, t_0) \tag{2.1}$$

Here and below, summation over k is carried out from 0 to ∞.

Using the relation, which holds for Chebyshev polynomials,

$$\frac{d}{d\xi}\sqrt{1 - \xi^2}U_n(\xi) = -(n + 1)\frac{T_{n+1}(\xi)}{\sqrt{1 - \xi^2}}$$

(where $T_n(\xi)$ is a Chebyshev polynomial of the first kind), we find the following series expansion for the derivative of the function $\bar{H}(\xi, t)$ in Chebyshev polynomials of the first kind

$$\frac{\partial}{\partial \xi}\bar{H}(\xi, t) = -H_0\sum(2k + 1)h_k(t)\frac{T_{2k+1}(\xi)}{\sqrt{1 - \xi^2}} \tag{2.2}$$

Using the formula for the pressure (1.5) and the integral relation[12]

$$\int_{-1}^{+1} \frac{T_n(y)dy}{\sqrt{1 - y^2}(y - x)} = \pi U_{n-1}(x)$$

we obtain the expression for the pressure in the form of a series expansion in Chebyshev polynomials of the second kind

$$\bar{P}(\xi, t) = H_0\sum(2k + 1)h_k(t)U_{2k}(\xi) \tag{2.3}$$

Hence, from the elasticity theory equations the relation between the magnitude of the opening $\bar{H}(\xi, t)$ and the pressure $\bar{P}(\xi, t)$, which is necessary for solving the balance equation, is obtained. The balance equation is an equation for the expansion coefficients $h_k(t)$ as well as for the unknown time-dependent function $R(t)$, which specifies the crack length and can be found from Eq. (1.6), which corresponds to the fracture condition.

Actually, when finding the expansion coefficients we must restrict ourselves to a finite number of terms. However, If we take into account that at the symmetry point $\xi = 0$ the flux changes sign, then in the case of a δ-like source shape the expression for the pressure has to contain a generalized function of the form $|\xi| = \xi\, sgn\, \xi$. Such a generalized function can be represented by an infinite series in Chebyshev polynomials of the first kind

$$\operatorname{sgn}\xi = \frac{4}{\pi}\sum \frac{(-1)^k}{2k+1} T_{2k+1}(\xi)$$

As a result, in the expression for the pressure (2.3) we can separate out an infinite subsequence by writing

$$\bar{P}(\xi, t) = H_0\{\sum (2k+1) h_k(t) U_{2k}(\xi) + B(t)(1 - \pi|\xi|)\} \tag{2.4}$$

and correspondingly

$$\bar{H}(\xi, t) = H_0\left\{\sqrt{1-\xi^2}\sum h_k(t) U_{2k}(\xi) + B(t) F(\xi)\right\},$$

$$\frac{\partial \bar{H}(\xi, t)}{\partial \xi} = -H_0\left\{\sum (2k+1) h_k(t) \frac{T_{2k+1}(\xi)}{\sqrt{1-\xi^2}} + B(t)\frac{dF(\xi)}{d\xi}\right\} \tag{2.5}$$

The function

$$F(\xi) = \frac{\xi^2}{2}\ln\frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}} \tag{2.6}$$

takes into account the contribution of the term on the right-hand side of Eq. (2.4) that is proportional to $B(t)$.

Note that the use of a series expansion for the value of the opening in Chebyshev polynomials of the second kind enables us to obtain a simple formula for the pressure (2.4) with the expansion coefficients $(2k + 1)h_k(t)$.

3. The self-similar solution

Self-similarity of solutions means that the solution can be represented in the form

$$H(x, t) = f(t)\bar{H}(x/R(t)) \tag{3.1}$$

whence it follows that (when there is no lag)

$$P(x, t) = (f(t)/R(t))\bar{P}(x/R(t)) \tag{3.2}$$

and from Eq. (1.6), which corresponds to the fracture condition, we obtain the relation

$$f^2(t)/R(t) = \text{const} \tag{3.3}$$

The self-similarity condition (3.1) means that all the expansion coefficients $h_k(t)$ and $B(t)$ must depend on time in the same manner, namely, $h_k(t) = f(t)a_k$ and $B(t) = f(t)b$, and the solution of the problem reduces to finding the functions $f(t)$ and $R(t)$, the expansion coefficients a_k and b for the given pumping intensity $q(t)$ and the initial crack volume V_0 (the specification of the initial values for the crack half-length $R_0 = R(t_0)$ and the value of the opening at the pumping point $H_0 = H(0, t_0)$ is incorrect since these quantities are not independent).

If the function $f(t)$ is normalized according to the condition $f(t_0) = 1$, the solution can be represented in the form

$$H(x, t) = H_0 f(t) h(\xi), \quad h(\xi) = \sqrt{1-\xi^2}\sum a_k U_{2k}(\xi) + bF(\xi)$$

$$P(x, t) = \frac{H_0 f(t)}{R(t)} p(\xi), \quad p(\xi) = \sum (2k+1) a_k U_{2k}(\xi) + b(1 - \pi|\xi|) \tag{3.4}$$

In this case $h(0) = 1$ and, taking into account the equalities $U_{2k}(0) = (-1)^k$ and $F(0) = 0$, we arrive at the requirement

$$\sum (-1)^k a_k = 1 \tag{3.5}$$

and from condition (1.6) it follows that $R(t) = R_0 f^2(t)$

The contribution to the mass balance equation for the injected fluid (1.4) from the term with the coefficient $B(t)$ in the braces in the representation of the solution for the pressure (2.4) gives a term proportional to $\delta(x)$, which compensates the corresponding expression on the right-hand side of the balance equation. The condition for the compensation of the singular terms gives

$$2\pi b H_0^4 f^4(t)/R^2(t) = q(t) \tag{3.6}$$

which, taking Eq. (3.3) into account, corresponds to the equality $q(t) = \text{const} = q_0$.

Calculation of the injected fluid volume using Eqs (1.7), (3.1) and (3.4) leads to the relation

$$V(t) = \int_{-R(t)}^{R(t)} H(x,t)dx = H_0 R_0 f^3(t) \int_{-1}^{+1} h(\xi)d\xi = V_0 f^3(t) \tag{3.7}$$

where

$$V_0 = \frac{\pi}{2}\alpha H_0 R_0, \ \alpha = a_0 - \frac{1}{3}b$$

Solving the equation

$$dV(t)/dt = q_0$$

we obtain

$$f(t) = \left[1 + \frac{q_0}{V_0}(t - t_0)\right]^{1/3} \tag{3.8}$$

One more relation between the parameters can be obtained after substituting the expression for the pressure $P(x, t)$ into fracture condition (1.6) and integrating. As a result, we get

$$R_0 = \pi^2\beta^2 H_0^2, \quad \beta = \sum(2k+1)a_k - b \tag{3.9}$$

and Eq. (3.6) takes the form

$$2b/(\pi^3\beta^4) = q_0 \tag{3.10}$$

Using the relations

$$U_{2k}(1) = 2k+1, \quad F(\xi) \to -\sqrt{1-\xi^2} \ \text{при} \ \ \xi \to 1$$

and the first equation of (3.4), we find that near the crack tip ($\xi \sim 1$)

$$h(\xi) \approx \beta\sqrt{1-\xi^2}, \quad \frac{dh(\xi)}{d\xi} \approx -\frac{\beta}{\sqrt{1-\xi^2}}$$

that, taking the identity $T_n(1) = 1$ into account, agrees with expansion (2.2). From this it follows that, when there is no lag, the gradient of the crack opening near the crack tip always has a singularity since $\beta \neq 0$ in view of Eq. (3.9).

4. Constructing the solution for the crack opening

It follows from the results obtained that solving the problem reduces to finding the expansion coefficients a_k and b and the parameters H_0 and R_0 for given values of the pumping rate q_0 and initial volume V_0, which are related by Eqs (3.7), (3.9) and (3.10). The initial values of the opening at the pumping point H_0 and the half-length of the fracture R_0 can be expressed in terms of the expansion coefficients a_k and b and the initial volume V_0 according to formulae

$$H_0 = \frac{1}{\pi}\left(\frac{2V_0}{\alpha\beta^2}\right)^{1/3}, \quad R_0 = \left(\frac{2\beta V_0}{\alpha}\right)^{2/3}$$

and the expansion coefficients can be found from the set of Eqs (3.5) and (3.10) and the balance Eq. (1.4), which, after using representations (3.4) and Eq. (3.8), takes the form

$$\frac{4b}{3\alpha}\left[h(\xi) - 2\xi\frac{dh(\xi)}{d\xi}\right] - \frac{d}{d\xi}\left[h^3(\xi)\frac{dp(\xi)}{d\xi}\right] = 2\pi b\delta(\xi) \tag{4.1}$$

Multiplying both sides of this equation by $U_{2k}(\xi)$ and integrating with respect to ξ between the limits from −1 to +1, we obtain an algebraic set of equations for the unknown coefficients a_k in terms of b, supplemented by the relation $\alpha = a_0 - b/3$. After integrating by parts and taking the boundary conditions $h(\pm 1) = 0$ into account, we get

$$2\pi(-1)^k b = 4bI_k^{(1)}/(3\alpha) + I_k^{(2)}$$

$$I_k^{(1)} = \int_{-1}^{+1}\left[U_{2k}(\xi) + 2\frac{d}{d\xi}(\xi U_{2k}(\xi))\right]h(\xi)d\xi, \quad I_k^{(2)} = \int_{-1}^{+1}\frac{dU_{2k}(\xi)}{d\xi}\left[h^3(\xi)\frac{dp(\xi)}{d\xi}\right]d\xi \tag{4.2}$$

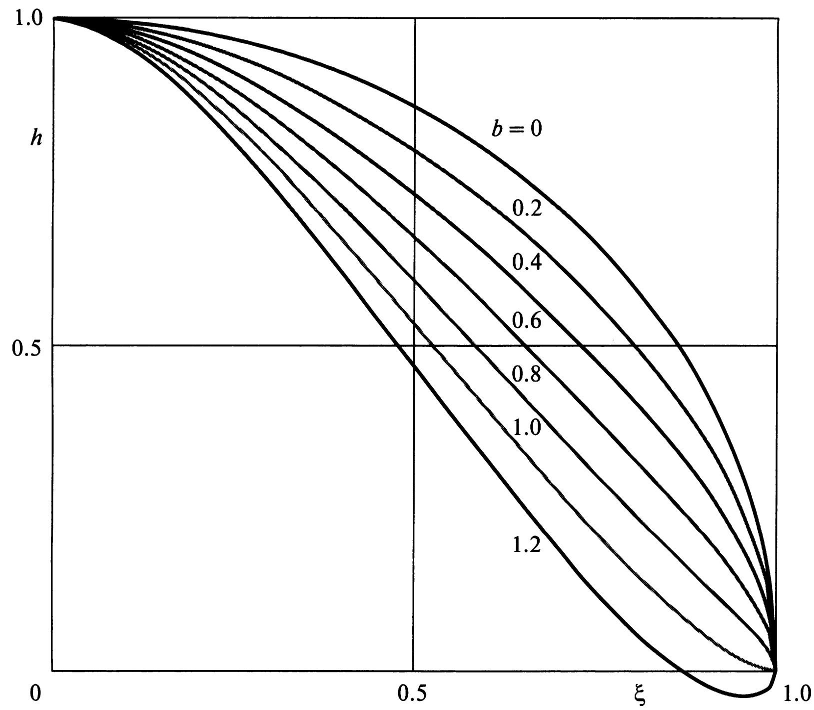
It can be seen that, when $k = 0$ and taking the equality $U_0(\xi) = 1$ into account, Eq. (4.2) is satisfied identically.

In the first approximation we assume that only the first term of the sum a_0 contributes to expansion (3.4), that is, all $a_k = 0$ if $k \neq 0$ and from condition (3.5) it follows that $a_0 = 1$. In this approximation the solution has the form

$$h(\xi) = \sqrt{1-\xi^2} + bF(\xi) \tag{4.3}$$

(the function $F(\xi)$ is defined by formula (2.6)).

Hence it follows that in this case the form of the solution is specified by single parameter b, which is defined by the pumping rate q_0. The form of the solution for various values of parameter b is shown in the figure; it can be seen that the solution is positive definite only when $b < 1$. If the pumping is slow ($q_0 < 0.3$), the value of parameter b is less then 0.2. One further restriction for the value of the parameter b follows from the requirement that the pressure must be positive, which, in the first approximation, gives $b < 1/(\pi - 1)$.



The second term in expansion (3.4) can be estimated by putting $a_1 \neq 0$ in this case $a_0 = 1 + a_1$, according to condition (3.5). The approximate solution turns out to be represented in the form

$$h(\xi) = \sqrt{1-\xi^2}(1 + 4a_1\xi^2) + bF(\xi) \tag{4.4}$$

Substituting this expression into Eq. (4.2) with $k = 1$, we obtain a non-linear algebraic equation for the unknown parameter a_1. Evaluation of the first integral on the left-hand side of Eq. (4.2) gives

$$I_1^{(1)} = \frac{\pi}{2}\left(4 + 11a_1 - \frac{16}{5}b\right)$$

When evaluating the second integral we will confine ourselves to taking into account only terms proportional to a_1 and b, assuming these parameters to be small, that is, in the expression for the flux we take

$$h(\xi) \approx \sqrt{1-\xi^2}, \quad \frac{dp(\xi)}{d\xi} = 24a_1\xi - \pi b \operatorname{sgn}\xi$$

As a result, we get

$$I_1^{(2)} \approx -\frac{6\pi}{5}a_1 - 2\pi b$$

Substituting the expression obtained into Eq. (4.2) we obtain a relation that enables us to estimate the role of the correcting term in the slow pumping approximation ($q_0 < 0.3$). We obtain

$$a_1 \approx -\frac{11}{90}b \tag{4.5}$$

At a higher pumping rate, when the value of b is not small compared to unity, we obtain a more complex set of algebraic equations for the coefficients a_k. This can be solved using numerical methods. Corresponding estimates show that even for $b \sim 1$, when the dimensionless pumping rate q_0 is not small, the expansion coefficients decay rapidly as k increases, and to describe the crack characteristics it is sufficient to use a finite number of terms in the representation of the crack opening in the form of a series expansion in Chebyshev polynomials of the second kind (3.4).

5. Conclusion

Within the framework of the Spence–Sharp model a procedure for constructing a self-similar solution, which determines the behaviour of hydraulic fracture characteristics, has been formulated. It has been shown that a self-similar solution can only be obtained for a constant rate of fluid pumping, and there are no other self-similar solutions of the problem. Note that Spence and Sharp[3] have obtained self-similar solutions of more general form for a power and exponential pumping law, due to the fact that they used an incorrect traditional statement of the problem when the local source at the symmetry point $x = 0$ was changed by specifying the flux at this point. In such a formulation the condition for compensating the singularities in the mass balance equation does not arise, and this leads to wider possibilities in specifying the time dependence of the flux at the symmetry point, whereas in a more correct statement the flux at this point equals zero (the flux distribution function changes sign).

The solution for the size of the crack opening has been represented in the form of a series expansion in Chebyshev polynomials of the second kind, and the coefficients of this expansion are solutions of a certain non-linear set of algebraic equations. It has been shown that at a not-too-high pumping rate the values of the expansion coefficients in the k-th mode decrease rapidly as k increases, and the use of a finite number of expansion coefficients is a good approximation. The laws of evolution of the length and maximum opening of the crack, as well as the relation between these quantities allowed by the conditions of the problem, have been found.

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